Journal of Hydrosciences and Environment
ISSN: 2345-5608

Numerical Modeling Based on Implicit Differential Quadrature Method for Two-Dimensional Groundwater Flow Problems

M.Adhami [1*], B.Kaya[2], A.Gharehbaghi[3] , H. Utku Çakır[4]

1, Ph.D of Department of Watershed Management Engineering, Faculty of Natural Resources, Tarbiat Modares University, Noor, Iran
2.Proffesor of Department of Civil Engineering, Faculty of Engineering, Dokuz Eylül University, Izmir, Turkey
3. Ph.D of Department of Civil Engineering, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, Türkiye
4. MSc of Department of Civil Engineering, Faculty of Engineering, Dokuz Eylül University, Izmir, Turkey

ARTICLEINFO

Article history:

Received: 14 Aug 2023
Accepted: 4 Oct 2023

Keywords:
Aquifer
Boussinesq equation
Finite difference method
Fluid flow

ABSTRACT

Predicting the behavior of fluid flow through the porous media is significantly important, especially in arid and semi-arid regions. In the present research, a novel numerical model based on the Implicit Differential Quadrature Method (IDQM) was developed to simulate the two-dimensional groundwater flow problems with the nonlinear Boussinesq equation. The data of the grid points were calculated from Chebyshev–Gauss–Lobatto formula. To evaluate the accuracy and the efficiency of the suggested model, three commonly reported in the literature groundwater problems were employed: (i) a problem related to the Boussinesq equation with an analytical solution, (ii) a hypothetical case of groundwater flow problem in a confined aquifer, and (iii) an aquifer pumping test. In addition, the results of the above-mentioned model were compared with the numerical predictions of the explicit and implicit schemes of the finite difference method (EFDM and IFDM). The findings demonstrated high accuracy, efficiency, and superiority of the numerical results of the IDQM. According to the results, the lowest values of root mean square error (RMSE) in the first, second, and third test cases with a range of (2.83E-23-5.65E-12), (0.072-0.065), and (0.230-0.100) were observed using IDQM. The values of dimensionless differences (DD) were also assigned the lowest values in this method in all tests. Briefly, the IDQM using less number of nodes provided more accurate results than EFDM and IFDM and as a result, is applicable to a wide range of problems in numerous fields of science and engineering.

1. Introduction

In the majority of the arid and semi-arid regions, groundwater is one of the vital resources supplying various domestic, agricultural, and industrial water requirements. Consequently, predicting the behavior of fluid flow through the porous media is significantly important. Due to the significance of this issue, various field and laboratory experiments have been carried out and numerous analytical and numerical solutions have been developed to investigate the aforementioned process (Hongen and Yang, 2012; Zolotukhin and Gayubov, 2019; Mohtashami et al., 2019; Mohtashami et al., 2022; Fo et al., 2023).

Some classical methods, e.g. finite difference method (FDM), finite element method (FEM), finite volume method (FVM) variational iteration method, and local boundary integral method are generally more popular in this domain (Hornberger et al. 1970; Igboekwe and Achi 2011; Tailor et al., 2013; Kovarik and Muzik, 2013; Bahrami et al., 2014; Liu et al., 2015; Gharehbaghi, 2017; Jasak & Uroić, 2020; Nyende et al., 2022; Liu et al., 2023 among many). But with dramatic developments that have occurred during recent decades in the field of computer science and mathematics, novel numerical techniques have been developed (Kaya, 2010; Gharehbaghi, 2016). Mostly, these new numerical methods have many advantages over the traditional techniques. For instance, in the most of groundwater flow problems studies the one-dimensional

*Corresponding author's email : m.adhami66@yahoo.com
DOI: 10.22111/jhe.2023.46400.1098

(1D) form is carried out (e.g., Kaya and Arisoy 2010, Tailor et al. 2013), however, in actual fact, this is a multidimensional phenomenon in nature and most of the time solving the 1D form of the problem is an oversimplification. In this regard, the adavantage of being a multidimentional method of DQM could solve such an issue which is more realistic and can satisfy most of natural problems. Additionally, mentioned method is capable to calculate more precise outcomes possessing fewer number of space and time nodes (Behroozi and Vaghefi, 2021; Gharehbaghi, 2022). Moreover, DQM accomplished simple solutions to achieve nonlinear equations without confusion and linearization. Scientists believe that in nonlinear problems DQM engages more points for functional value estimation of a specific point. This procedure facilitates quick converging and more reliable results in comparison with FEM or FDM, (Chen and Zhong, 1997; Gharehbaghi 2022). Briefly, the traditional numerical techniques require relatively intricate mathematical exploitations and apply sophisticated calculations; in contrast, DQM could produce outcomes at high accuracy, using a small grid points, little calculation labor and the flexibility in choosing the grid points" (Ciftci, 2017).

In this domain, Hornberger et al. (1970) solved the Boussinesq equation with the FDM to simulate the problems of groundwater flow and groundwater recession in response to fluctuations in the flow stage. Larabi and Smedt (1997) developed a numerical procedure for a stationary flow using a FEM with a fixed mesh. They assumed that the flow was considered to occur only in the saturated freshwater part. Igboekwe and Achi (2011) suggested a numerical model based on the FDM for groundwater flow problems to calculate flow direction, flow rates, and hydraulic heads through an aquifer. Kaya and Arisoy (2011) employed the explicit form of the 1D DQM to solve the aquifer flow problems. In order to examine the results, they used three well-known analytical expressions as follows: (i) a confined aquifer flow with time-dependent boundary conditions, (ii) a composite confined aquifer, and (iii) an unconfined aquifer with seepage. Ameli and Abedini (2016) explored the performance of high-order differential quadrature (DQM) and radial basis function (RBF)-DQM methods. Thet also compared mentioned methods with low-order finite difference and finite element methods. They used Theis' exact analytical solution and considered head data as benchmarks. Their findings showed efficiency of high-order DQM and RBF-DQM compared to low-order numerical methods in simulating flow by a pumping well. Baghlani and Nikzad (2018) investigated DQM's applicability to solve Richards'equation for Water flow in porous media. For this aim, they used two constitutive models named White and Broadbrige and Van Genuchten. The results demonstrated robustness of DQM which provides excellent results considering numerical techniques. Behroozi and Vaghefi (2022) conducted Thin Plates Spline-Based Differential Quadrature Method(TPS-DQM) as a numerical method in a groundwater field. High accuracy and sufficient reliability of mentioned method was introduced as findings.

According to the literature review outlined in the previous lines, most of the researchers have been interested to handle the groundwater flow problems with one of the classical methods or in 1D conditions. The main aim of this study is to establish a novel, robust and efficient 2D numerical model with DQM for solving the time-dependent Boussinesq equation to predict the behavior of groundwater flow in various conditions. Moreover, another difference of the present study is that despite other researchers who mostly have used the explicit form of DQM, to achieve more reliable results the implicit scheme IDQM is considered. Besides, the results of the proposed model are compared with the numerical predictions of the explicit and implicit schemes of the finite difference method (EFDM and IFDM) which is essential in groundwater simulation. Three groundwater flow problems were employed to verify the efficiency of the introduced model. In the first test case, an analytical solution introduced by Akman (2003) was used to solve the 2D transient groundwater flow problem (Boussinesq equation). In the second test case, a hypothetical test suggested by Wang and Anderson (1995) was selected to predict the groundwater flow in a confined aquifer with given initial and boundary conditions. As a last numerical experiment, a collection of empirical data provided by Şeber (1964) was utilized to model the aquifer pumping test problem.

2. Materials and Methods

2.1. Governing Equations

Like other engineering problems, the groundwater flow phenomenon can be expressed with mathematical equations. The mathematical definition of groundwater flow has been extracted from a combination of conservation of mass and Darcy's law. In a horizontal confined aquifer with the thickness of b, the mathematical definition of groundwater flow is given as follows: (Wang and Anderson, 1995)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t} \tag{1}$$

where h, S, T, and t are the piezometric level, storage coefficient, transmissivity, and time, respectively. Furthermore, the parameters of x and y are the directions in a Cartesian coordinate system. However, in the case of existing recharge points, the following equation can be used. (Wang and Anderson, 1995)

$$\frac{d^2 h}{dx^2} + \frac{d^2 h}{dy^2} = \frac{S}{T}\frac{dh}{dt} - \frac{R(x,y,t)}{T} \tag{2}$$

where, R(x,y,t) is the rate of flow (discharge or recharge).The mathematical expression of unconfined groundwater flow in an isotropic and homogeneous aquifer is presented as follows:

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)=\frac{S_y}{K}\frac{dh}{dt} \quad (3)$$

where Sy and K are the specific yield (dimensionless), and constant hydraulic conductivity, respectively. Eq. (3) is known as the nonlinear Boussinesq equation. Due to its nonlinearity, the analytical solution of Eq. (3) is usually difficult to obtain except for some special cases (Liang and Zhang 2011). The linearization of Eq. (3) is actually based on the Dupuit-Forchheimer assumption (Todd, 1980; Ku and Tsai, 2013). In the present paper, the following linearized Boussinesq equation is applied to simulate the case of existing recharge points:

$$\frac{K}{2}\left(\frac{\partial^2 h^2}{\partial x^2}+\frac{\partial^2 h^2}{\partial y^2}\right)=S_y\frac{\partial h}{\partial t}-R(x,y,t) \quad (4)$$

where by using the transformation of $v=h^2$ the following equation can be extracted (Kaya and Arısoy, 2010)

$$\frac{\mathrm{K}}{2}\left(\frac{\partial^2 \mathrm{v}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{v}}{\partial \mathrm{y}^2}\right)=\frac{\mathrm{S}}{2\sqrt{\mathrm{v}}}\frac{\partial \mathrm{v}}{\partial \mathrm{t}}-R(x,y,t) \quad (5)$$

It is important to mention that in the implicit solutions which will be appeared in the next sections, by using the old-time values of the expression $\sqrt{v}$, a semi-implicit solution will be obtained.

2.2. Numerical Scheme

In this research, a numerical model is proposed and analyzed for the first time with the IDQM for 2D groundwater flow problems. Moreover, the EFDM and IFDM are used to verify the reliability of the developed model. More details about these numerical techniques, implementations, and computational results are presented in the next sections.

2.3. The Differential Quadrature Method (DQM)

As stated earlier in this paper, the IDQM is used to establish a new 2D numerical model. For the first time, Bellman and Casti in 1972 were suggested this technique. Subsequently, the aforementioned method is applied to the wide range of engineering problems and many researchers tried to improve it (e.g., Quan and Chang 1989, Shu and Richards 1992, Shu 2000, Bert and Malik, 1996, Kaya 2010, Kaya and Arisoy 2010, Chan et al. 2014, Kurtaran 2015, Gharehbaghi 2016). The key idea of this technique is based on Gauss Quadrature method. "Extending Gauss quadrature to finding the derivatives of various orders of a differentiable function gives rise to differential quadrature" (Bellman and Casti 1971). "In other words, the derivatives of a function are approximated by weighted sums of the function values at a group of nodes" (Zong and Zhang 2009).

$$\begin{Bmatrix} u'(x_1) \\ u'(x_2) \\ \vdots \\ u'(x_n) \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & . & A_{1n} \\ A_{21} & A_{22} & . & A_{2n} \\ . & . & . & . \\ . & . & . & . \\ A_{n1} & A_{n1} & . & A_{nn} \end{bmatrix} \begin{Bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_n) \end{Bmatrix} \quad (6)$$

where xj, A(r)ij and u(xj) are the nodes, the weighting coefficients of the rth-order derivatives, and the function values at discrete points, respectively. Functional approach is required to obtain the values of weighting coefficients. In order to achieve the desired accuracy with DQM, calculating the appropriate values of weighting coefficients is inevitable. Thus, various formulas have been suggested to determine these coefficients. In this study, the formulas developed by Quan and Chang (1989), which are based on Lagrange polynomial, are employed as follows (Shu, 2000):

$$j\neq i: \quad A_{ji}^{(1)}=\frac{\prod(x_i)}{(x_i-x_j)\prod(x_j)} \quad (7)$$

$$j=i: \quad A_{ii}^{(1)}=\sum_{j\neq i}\frac{1}{(x_i-x_j)} \quad (8)$$

In addition, the values of high-order weighting coefficients can be calculated as below;

$$A_{ji}^{(r)}=r\left(A_{ji}^{(1)}A_{ii}^{(r-1)}+\frac{A_{ji}^{(r-1)}}{x_j-x_i}\right) \quad (9)$$

$$A_{ii}^{(r+1)}=-\sum_{j\neq i}\left(A_{ji}^{(r+1)}\right) \quad (10)$$

Also, an implicit backward difference scheme is used to discretize the governing equation in time. Therefore, by some replacement and manipulations, the solution of Eq. (2) with the implicit differential quadrature method is obtained as follows:

$$\begin{aligned} &-\frac{\mathrm{S}}{\mathrm{T}\Delta\mathrm{t}}h_{i,m}^{t}+\sum_{j=2}^{Nx-1}A_{j,i}^{(2)}h_{j,m}^{t} \\ &+\sum_{n=2}^{Ny-1}B_{n,m}^{(2)}\,h_{i,n}^{t} \\ &=-\left(\frac{\mathrm{S}}{\mathrm{T}\Delta\mathrm{t}}h_{i,m}^{t-\Delta t}+A_{1,i}^{(2)}h_{1,m}^{t}+A_{nx,i}^{(2)}h_{nx,m}^{t}\right. \\ &+\ B_{1,m}^{(2)}h_{i,1}^{t}+B_{ny,m}^{(2)}h_{i,ny}^{t} \\ &\left.+\frac{R(i,m,t)}{T}\right) \end{aligned} \quad (11)$$

where B is the weighting coefficient for the derivative in y direction. It is worthwhile to mention that by some manipulation, it is possible to extract the values of weighting coefficients for a 2D solution from 1D solution. As a final point, scientists have provided several methods

to calculate the place of nodes. "It can be selected with equal intervals, or non-equal intervals like Chebyshev–Gauss–Lobatto grid points, or with the normalization of the routes of Legendre polynomials" (Shu 2000, Kaya 2010, Gharehbaghi 2016). In this study, the formula proposed by Chebyshev-Gauss-Lobatto for grid points is applied as follows:

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N_x - 1}\pi\right)\right] \quad i = 1,2,\dots,N_x \tag{12a}$$

$$y_j = \frac{1}{2}\left[1 - \cos\left(\frac{j-1}{N_y - 1}\pi\right)\right] \quad j = 1,2,\dots,N_y \tag{12b}$$

2.4. The finite difference method (FDM)

The FDM is a popular technique for discretizing ordinary and partial differential equations. To compare the numerical results of the developed model, the explicit and implicit forms of FDM are solved, too. In addition to the discretization of the space, for time discretization an implicit backward difference scheme is enforced. Thus, by using the second order central finite difference formula as follows:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} \tag{13}$$

The final forms of explicit finite difference method and implicit finite difference method can be written as follows:

$$\begin{aligned} h_{i,j}^{t} &= \frac{T\Delta t}{S}\frac{h_{i+1,j}^{t-\Delta t} - 2h_{i,j}^{t-\Delta t} + h_{i-1,j}^{t-\Delta t}}{(\Delta x)^2} \\ &+ \frac{T\Delta t}{S}\frac{h_{i,j+1}^{t-\Delta t} - 2h_{i,j}^{t-\Delta t} + h_{i,j-1}^{t-\Delta t}}{(\Delta x)^2} + h_{i,j}^{t-\Delta t} \\ &+ \frac{\Delta t}{S}R(x,y,t) \end{aligned} \tag{14}$$

$$\begin{aligned} &\left(\frac{1}{(\Delta x)^2}\right)h_{i+1,j}^{t} \\ &+ \left(\frac{1}{(\Delta y)^2}\right)h_{i,j+1}^{t} \\ &- \left(\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2} + \frac{S}{T\Delta t}\right)h_{i,j}^{t} + \left(\frac{1}{(\Delta x)^2}\right)h_{i-1,j}^{t} \\ &+ \left(\frac{1}{(\Delta y)^2}\right)h_{i,j-1}^{t} = -\frac{S}{T}\frac{h_{i,j}^{t-\Delta t}}{\Delta t} - \frac{R(x,y,t)}{T} \end{aligned} \tag{15}$$

Based on our best knowledge of this subject, the majority of the studies related to FDM have been employed the structured mesh. Therefore, we decided to develop a structured mesh system for both of the EFDM and IFDM.

3.Results and discussion

To investigate the behavior of groundwater flow, three test cases well-documented in the literature are applied: (i) a problem related to the transient diffusion equation with an analytical solution, (ii) a hypothetical case of groundwater flow problem in a confined aquifer, and (ii) an aquifer pumping test. To compare and analyze the numerical results of the developed model values of dimensionless difference (DD) and root mean square error (RMSE) are determined.

3.1. Test case 1

In the first numerical experiment, the following equation is used to predict the 2D transient groundwater flow:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{\partial h}{\partial t} \tag{16}$$

Akman (2003), derived the following analytical expression to calculate this equation

$$h(x,y,t) = sin(\pi x)sin(2\pi y)e^{-5\pi^2 t} \tag{17}$$

The initial and boundary conditions of this problem are given as below. Also, the illustration of the implemented initial and boundary conditions is demonstrated in Fig. (2). In this test case, for the piezometric level (h), the following sinusoidal expression is assumed to satisfy the initial conditions.

$$h(x,y,0) = sin(\pi x)sin(2\pi y) \tag{18}$$

Based on the Dirichlet boundary condition, which is presented below, the value of h, in both of the x and y directions, is taken as zero.

$$h(0,y,t) = 0 \;\; h(1,y,t) = 0$$

Dirichlet boundary condition (19)

$$h(x,0,t) = 0 \;\; , h(x,1,t) = 0$$

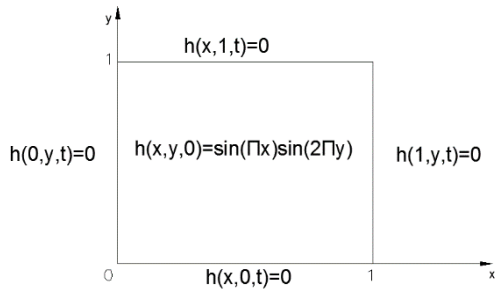


Fig. 1. The illustration of implemented initial and boundary conditions in the first test case

Different numerical scenarios are constructed to obtain the comprehensive evaluation of the model. In this regard, the model is run with five different time intervals (Δt=0.0001, 0.0005, 0.001, 0.005, 0.01) and four different node numbers (Nx=Ny=9, 13, 17, 21). The simulation life time is selected as 10. It is important to note that in tables and figures below the x, y, Nx, Ny, h, Nt, Δt, maxDD, Exact and Measurement are the abbreviations of distance along x axis, distance along y axis, number of node points along x axis, number of node points along y axis, the piezometric level, number of time intervals, the value of implemented time intervals, the max values of DD, result of the analytical solution, and the observed values reported in literature, respectively. In order to make a better comparison between the predicted results of the imposed schemes in developed model, the DD and RMSE values are calculated. The above-mentioned values are computed by the following equations.

$$DD = \frac{\left|\max\left(h_{numerical\ solution} - h_{analytical\ solution}\right)\right|}{\max\left(h_{analytical\ solution}\right)} \quad (20)$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N_x N_y}\left(h_{numerical\ method} - h_{analytic\ solution}\right)^2}{N_x N_y}} \quad (21)$$

According to the numerical outcomes of Eqs. (20) and (21) shown in table (1), it is clearly observed that the predicted results of IDQM are significantly more accurate and reliable than the numerical outcomes of EFDM and IFDM. Meanwhile, it can be seen that because of stability issues the EFDM is generally not able to initiate the simulation process. Based on Adak (2020), EFDM does not always give stable solutions. In other words, this method is conditionally stable.

With regard to data revealed in the table (1), it is possible to assess the influence of time intervals and node numbers on the results of the 2D proposed model. Based on these data, increasing the number of nodes in x and y directions or decreasing the time intervals having a very tiny influence on the precision of numerical outcomes of IDQM and the improvements are negligible, i.e., the accuracy and sensitivity of numerical results are independent from number of node points or time intervals. Close agreement was found with the outcomes of Behroozi and Vaghefi (2022). Meanwhile, based on the results of execution time presented in table (1), it may be observed that the execution time of IFDM and IDQM are so close to each other. Thus, it can be said that by using the IDQM, it is possible to obtain more reliable results in the same execution time as IFDM. It must be noted that in this numerical experiment, the analytical solution can obtain the exact values in every required point, thus, the node points of numerical methods are employed to calculate the values of desired differences. Meanwhile, the error values are calculated by using these points, too. In this step, in order to make a clearer picture of the results, only some of the contour plots of differences between numerical predictions of the implemented methods and analytical solution are shown in Figs. (2) - (4).

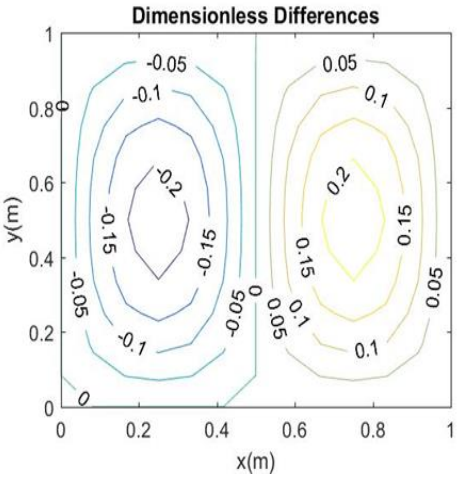


Fig. 2. The values of EFDM-Exact in N_x=N_y=13 and N_t=1001

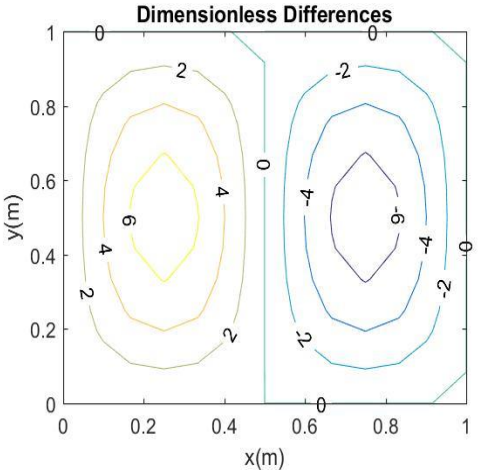


Fig. 3. The values of IFDM-Exact in N_x=N_y=13 and N_t=1001

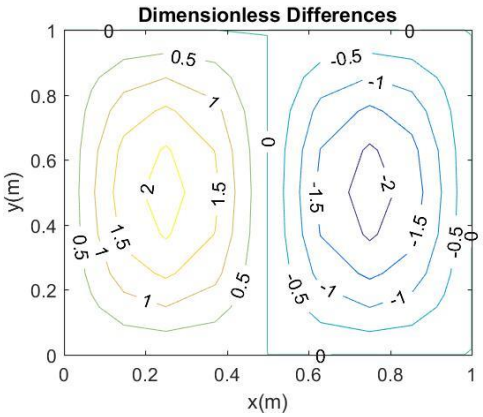


Fig. 4. The values of IDQM-Exact in N_x=N_y=13 and N_t=1001

As mentioned in the previous lines, the data in table (1) revealed that by applying the small number of nodes in x and y directions, EFDM generally cannot obtain the results. About this table it is important to note that the calculated values in point x=0.5 and y=0.25 are used to determine the values of column RMSE (in x=0.5, y=0.25). On the other hand, in order to calculate the values of column RMSE (at tmax), the errors of all of the nodes are employed. The values of piezometric level at the time of tmax are very closed to the steady state condition and, therefore, the error values are extremely small. In conclusion, the values of RMSE in column RMSE (at tmax) are much smaller than the values of RMSE in column RMSE (in x=0.5, y=0.25).

3.2.Test case 2

In the second test case, the 2D groundwater flow problem in a confined aquifer with given initial and boundary conditions is handled. In this problem, a draw well is considered in the middle of the area, with the constant flow rate of water. This system can be expressed by the equation (2). The draw well is located in the x=2000 (m), and y=2000 (m) coordinates. As emphasized previously, the flow rate is constant. The initial conditions of the piezometric level (h), is constant and can given as below;

$$h(x,y,0) = 10 \tag{22}$$

$$m_{i,j}^{well} \geq 2000 \longrightarrow h(x,y,t) = 10 \tag{23}$$

The illustration of properly imposed initial and boundary conditions, is presented in Fig. (5).

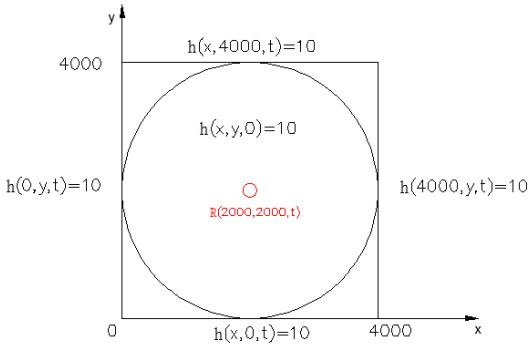


Fig. 5. The illustration of implemented boundary conditions in the second test case

In this groundwater flow problem, the aquifer coefficients are T=300 m2/day and S=0.002. The well is discharged (R) at a constant rate of 2000 m3/day. For exact solution of this test case, Thiem equation is employed in steady state (Wang and Anderson, 1995). Thiem equation could be expressed as follow:

$$h_r = h_{re} + \frac{Q}{2\pi T} ln\,(\frac{r}{re}) \tag{24}$$

In this equation, r and re are the distances to the source and re indicates the point where the groundwater level is known. Accordingly, the numerical results of the model are compared with Thiem equation results. The numerical results of DD and RMSE along with required execution time for each technique in the second test case are demonstrated in Table (2). Furthermore, the illustration of the results of numerical analysis for Nx=Ny=51 and Nt=4001 might be seen in Fig. (6) and the results of differences between the Thiem solution and the employed approaches for the same node number and time intervals are shown in Figs. (7) - (9). Similar to the first test case, since the desired value can be easily calculated, the amounts of RMSE and DD are calculated in the node points applied by numerical methods. The outcomes revealed that the numerical results of IDQM are more precise, reliable and convenient than the other two methods. As a final point in this case it is obvious that the required time for computational process in IDQM is not longer than IFDM. Actually, in this numerical experiment, the required time for IDQM is slightly smaller than IFDM. These results of Ghahri and Meraji (2011) approve this outcome.

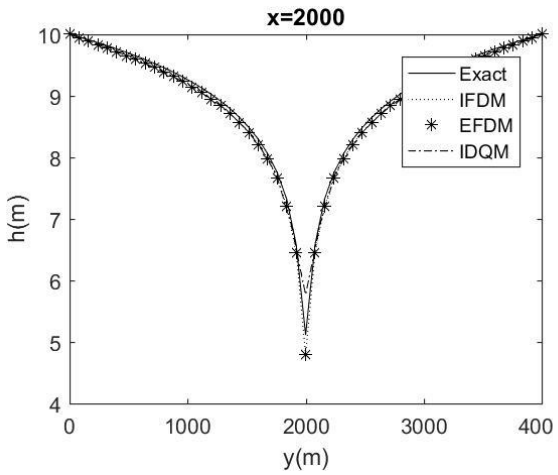


Fig. 6. The results of exact, EFDM, IFDM, and IDQM in N_x=N_y=51 and Nt=4001

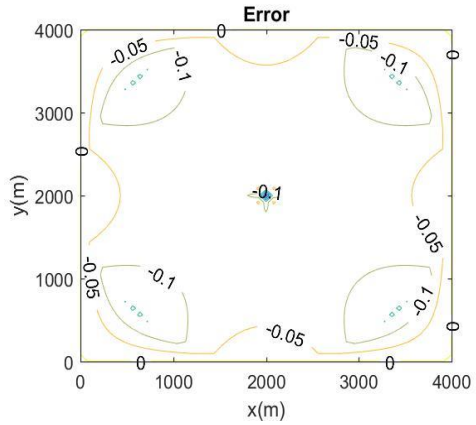


Fig. 7. The values of EFDM-Exact in N_x=N_y=51 and Nt=4001

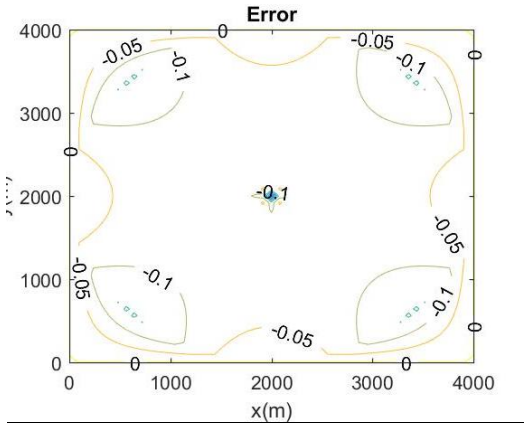


Fig. 8. The values of IFDM-Exact in N_x=N_y=51 and Nt=4001

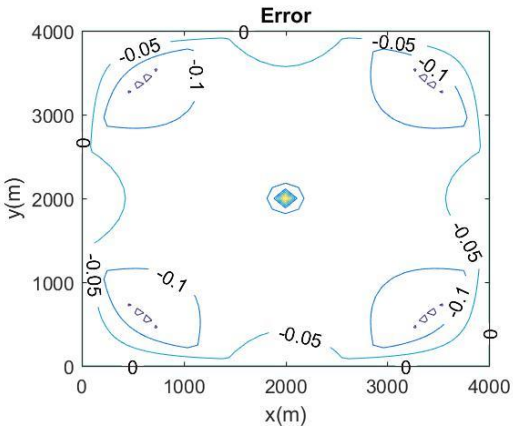


Fig. 9. The values of IDQM-Exact in Nx=Ny=51 and Nt=4001

3.3.Test case 3

In this test case, the proposed model is evaluated by using a well-documented aquifer-pumping test conducted by Şeber (1964). The above-mentioned field experiment was carried out in Kızılçubuk research draw well located in the Erbaa plain in Turkey. This aquifer located between 11 and 70 meters depth, i.e., thickness is 59 m, and the implemented pumping flow rate (R) was 33.9 l/s. The values of transmissivity (T) and storage coefficient (S) was estimated to be 354.7 m3/day/m and 0.031, respectively, for Eq. (2). In the initial and boundary conditions, calculations were made taking the value of the h-level descent to zero. The results of DD and RMSE for various node points and time intervals for EFDM, IFDM and IDQM are presented in Table (3). Additionally, to better understand the differences that arise between the numerical predictions of DD and water levels, the results for two different node points (Nx=Ny= 15, Nt=7201 and Nx=Ny= 21, Nt=721) are demonstrated in Figs. (10) - (13). According to these outcomes, it can be obviously seen that the IDQM always obtain more accurate results. Although the differences between EFDM and IFDM are so tiny, but between these two methods the results of IFDM are more reliable than EFDM.

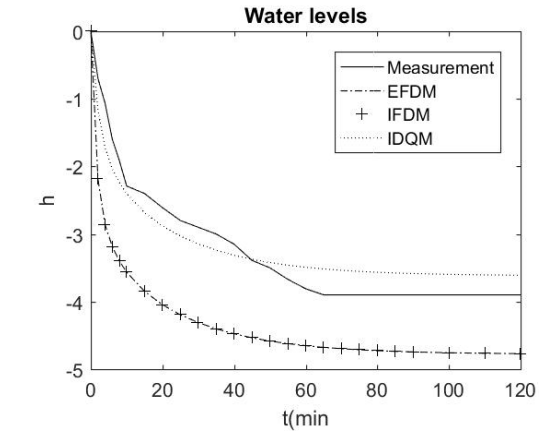


Fig. 10. The water levels in Nx=Ny=15 and Nt=7201 (Şeber, 1964)

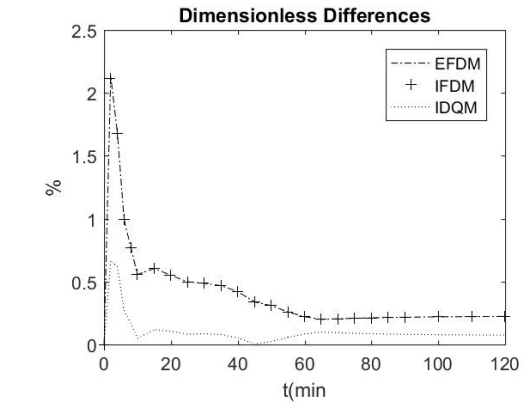


Fig. 11. The values of dimensionless differences in Nx=Ny=15 and Nt=7201

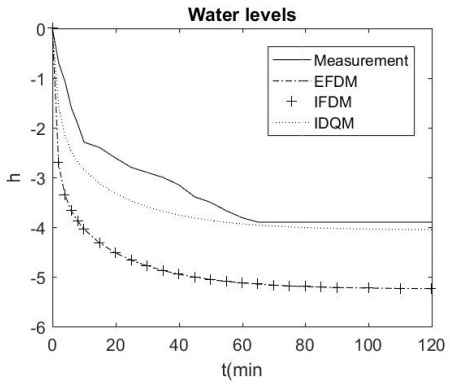


Fig. 12. The water levels in Nx=Ny=21 and Nt=721 (Şeber, 1964)

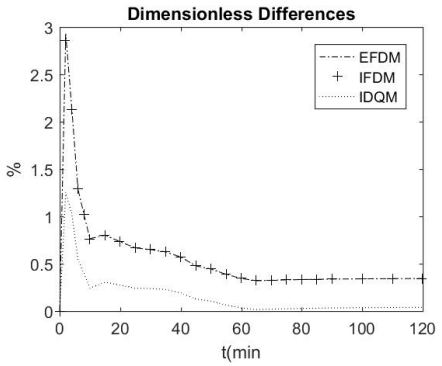


Fig. 13. The values of dimensionless differences in Nx=Ny=21 and Nt=721

4. Conclusion

This paper documented the development of a new numerical model based on the implicit scheme of the DQM that solved the time-dependent nonlinear Boussinesq equation to simulate 2D groundwater flow problems. To assess the accuracy of established model, three groundwater flow problems reported in the literature were used: (i) a problem related to the transient diffusion equation with an analytical solution, (ii) a hypothetical case of groundwater flow problem in a confined aquifer, and (ii) an aquifer-pumping test. Additionally, the results of the introduced model were verified numerically with explicit and implicit forms of FDM. Meanwhile, to make a better comparison between the predicted outcomes of the suggested model the DD and RMSE values were calculated. In all of the numerical methods discussed previously, to discretize the time parameter in the groundwater flow equation an implicit backward difference scheme was used. Also in the EFDM and IFDM, the second order central finite difference formula was implemented. Results from all of the test cases were indicated that in the same number of node points and time intervals, IDQM was achieved to obtain the most accurate results. Meanwhile, it may be observed that the execution time of IFDM and IDQM were so close to each other. Therefore, by using the IDQM it is generally possible to obtain more reliable results in the same execution time of IFDM. Owing to the comprehensive analysis of numerical predictions in this study, the IDQM obtain more accurate and reliable results than EFDM and IFDM. According to the results presented in this paper, DQM with using less number of nodes provided more accurate results than FDM. Eventually, the numerical results confirm that the DQM is applicable to a wide range of problems in numerous fields of science and engineering.

Table 1. The values of DD and RMSE for the first test case

Method	N_x	N_y	N_t	RMSE (at tmax)	maxDD	Δt	Execution time (sec)	RMSE (in x=0.5, y=0.25)
EFDM	9	9	21	-	-	-	-	-
IFDM	9	9	21	1.31E-11	7.95E+10	5.00E-02	0.35	0.05003
IDQM	9	9	21	5.18E-12	4.28E+10	5.00E-02	0.37	0.04573
EFDM	13	13	21	-	-	-	-	-
IFDM	13	13	21	9.64E-12	5.64E+10	5.00E-02	0.36	0.04874
IDQM	13	13	21	5.39E-12	4.28E+10	5.00E-02	0.37	0.04772
EFDM	21	21	21	-	-	-	-	-
IFDM	21	21	21	8.34E-12	4.73E+10	5.00E-02	0.52	0.04809
IDQM	21	21	21	5.56E-12	4.28E+10	5.00E-02	0.53	0.04709
EFDM	31	31	21	-	-	-	-	-
IFDM	31	31	21	8.01E-12	4.47E+10	5.00E-02	1.78	0.04762
IDQM	31	31	21	5.65E-12	4.28E+10	5.00E-02	2.11	0.04753
EFDM	9	9	101	-	-	-	-	-
IFDM	9	9	101	7.03E-18	4.27E+04	1.00E-02	0.72	0.01828
IDQM	9	9	101	1.24E-18	1.03E+04	1.00E-02	0.74	0.01255
EFDM	13	13	101	-	-	-	-	-
IFDM	13	13	101	3.32E-18	1.94E+04	1.00E-02	0.81	0.01668
IDQM	13	13	101	1.29E-18	1.03E+04	1.00E-02	0.81	0.01543
EFDM	21	21	101	-	-	-	-	-
IFDM	21	21	101	2.28E-18	1.29E+04	1.00E-02	1.44	0.01588
IDQM	21	21	101	1.33E-18	1.03E+04	1.00E-02	1.41	0.01466
EFDM	31	31	101	-	-	-	-	-
IFDM	31	31	101	2.04E-18	1.14E+04	1.00E-02	7.4	0.01554
IDQM	31	31	101	1.36E-18	1.03E+04	1.00E-02	7.61	0.01548
EFDM	9	9	201	-	-	-	-	-
IFDM	9	9	201	1.71E-19	1037	5.00E-03	0.89	0.01118
IDQM	9	9	201	2.27E-20	187.7	5.00E-03	0.88	0.00682
EFDM	13	13	201	-	-	-	-	-
IFDM	13	13	201	6.91E-20	404.5	5.00E-03	1.03	0.009512
IDQM	13	13	201	2.37E-20	188	5.00E-03	1.02	0.008197
EFDM	21	21	201	-	-	-	-	-
IFDM	21	21	201	4.37E-20	248	5.00E-03	2.3	0.00867
IDQM	21	21	201	2.44E-20	188	5.00E-03	2.27	0.00747
EFDM	31	31	201	-	-	-	-	-
IFDM	31	31	201	3.81E-20	212.7	5.00E-03	13.79	0.00836
IDQM	31	31	201	2.48E-20	188	5.00E-03	13.8	0.00828
EFDM	9	9	1001	2.66E-22	1.618	1.00E-03	1.71	0.00441
IFDM	9	9	1001	3.86E-21	23.44	1.00E-03	2.03	0.00484
IDQM	9	9	1001	2.72E-22	2.264	1.00E-03	2.02	0.00434
EFDM	13	13	1001	3.92E-23	0.2298	1.00E-03	1.81	0.00037
IFDM	13	13	1001	1.20E-21	7.043	1.00E-03	2.64	0.00310
IDQM	13	13	1001	2.83E-23	2.263	1.00E-03	2.56	0.00037
EFDM	21	21	1001					
IFDM	21	21	1001	6.19E-22	3.513	1.00E-03	8.61	0.002223
IDQM	21	21	1001	2.92E-22	2.26	1.00E-03	8.59	0.001251
EFDM	31	31	1001					
IFDM	31	31	1001	4.95E-22	2.763	1.00E-03	63.13	0.001939
IDQM	31	31	1001	2.97E-22	2.27	1.00E-03	63.87	0.001857

Table 2. The values of DD and RMSE for the second test case

Method	N_x	N_y	N_t	RMSE	maxDD	Δt	Execution time(sec)
EFDM	13	13	41	-	-	-	-
IFDM	13	13	41	0.076519	0.32041	1	0.52
IDQM	13	13	41	0.071814	0.161832	1	0.55
EFDM	21	21	41	-	-	-	-
IFDM	21	21	41	0.077618	0.322451	1	0.8
IDQM	21	21	41	0.066668	0.161218	1	0.83
EFDM	31	31	41	-	-	-	-
IFDM	31	31	41	0.078061	0.323083	1	3.17
IDQM	31	31	41	0.065252	0.172327	1	3.65
EFDM	13	13	401	0.076814	0.321185	0.1	1
IFDM	13	13	401	0.076769	0.321068	0.1	1.47
IDQM	13	13	401	0.071977	0.16243	0.1	1.44
EFDM	17	17	401	0.077467	0.322572	0.1	1.01
IFDM	17	17	401	0.077423	0.322456	0.1	2.23
IDQM	17	17	401	0.068404	0.158919	0.1	2.17
EFDM	21	21	401	-	-	-	-
IFDM	21	21	401	0.077869	0.323095	0.1	3.98
IDQM	21	21	401	0.066834	0.160577	0.1	3.91
EFDM	13	13	801	0.076803	0.321156	0.05	1.52
IFDM	13	13	801	0.076781	0.321098	0.05	2.17
IDQM	13	13	801	0.071985	0.162458	0.05	2.1
EFDM	21	21	801	0.077903	0.323181	0.05	1.53
IFDM	21	21	801	0.077881	0.323124	0.05	7.19
IDQM	21	21	801	0.066841	0.160548	0.05	7.07
EFDM	31	31	801	-	-	-	-
IFDM	31	31	801	0.078327	0.323752	0.05	55.46
IDQM	31	31	801	0.065427	0.17166	0.05	53.95
EFDM	13	13	4001	0.076794	0.321133	0.01	4.6
IFDM	13	13	4001	0.07679	0.321121	0.01	7.83
IDQM	13	13	4001	0.07199	0.162479	0.01	7.67
EFDM	21	21	4001	0.077894	0.323159	0.01	5.29
IFDM	21	21	4001	0.07789	0.323147	0.01	33.92
IDQM	21	21	4001	0.066847	0.160525	0.01	31.91
EFDM	31	31	4001	0.07834	0.323786	0.01	6.64
IFDM	31	31	4001	0.078336	0.323775	0.01	278.22
IDQM	31	31	4001	0.065434	0.171637	0.01	271.99
EFDM	51	51	4001	0.078974	0.324106	0.01	11.65
IFDM	51	51	4001	0.07897	0.324095	0.01	4031.11
IDQM	51	51	4001	0.065067	0.178204	0.01	4028.51

Table 3. The values of DD and RMSE for the third test case

Method	N_x	N_y	N_t	RMSE	maxDD	Δt	Execution time(sec)
EFDM	11	11	7201	0.33283	1.27126	1.16E-05	12.14
IFDM	11	11	7201	0.332225	1.258236	1.16E-05	15.85
IDQM	11	11	7201	0.230643	0.219468	1.16E-05	15.7
EFDM	15	15	7201	0.388462	2.133659	1.16E-05	13.72
IFDM	15	15	7201	0.388023	2.117234	1.16E-05	24.67
IDQM	15	15	7201	0.104229	0.66691	1.16E-05	23.42
EFDM	21	21	7201	0.386815	2.949797	1.16E-05	16.52
IFDM	21	21	7201	0.386542	2.935818	1.16E-05	66.26
IDQM	21	21	7201	0.12754	1.316566	1.16E-05	64
EFDM	31	31	7201	0.345886	3.773254	1.16E-05	23.67
IFDM	31	31	7201	0.345722	3.761614	1.16E-05	494.43
IDQM	31	31	7201	0.160166	2.069969	1.16E-05	488.34
EFDM	11	11	3601	0.3326	1.263391	2.32E-05	5.61
IFDM	11	11	3601	0.331392	1.237252	2.32E-05	7.39
IDQM	11	11	3601	0.23067	0.215158	2.32E-05	7.35
EFDM	15	15	3601	0.388325	2.128136	2.32E-05	6.06
IFDM	15	15	3601	0.387443	2.094921	2.32E-05	11.35
IDQM	15	15	3601	0.103774	0.652242	2.32E-05	11.12
EFDM	21	21	3601	0.386725	2.944833	2.32E-05	7.29
IFDM	21	21	3601	0.386176	2.916455	2.32E-05	33.24
IDQM	21	21	3601	0.127114	1.300542	2.32E-05	30.7
EFDM	31	31	3601	0.345826	3.768045	2.32E-05	8.92
IFDM	31	31	3601	0.345497	3.744466	2.32E-05	244.61
IDQM	31	31	3601	0.159915	2.053879	2.32E-05	266.48
EFDM	11	11	1441	0.331825	1.239513	5.79E-05	2.53
IFDM	11	11	1441	0.328804	1.173497	5.79E-05	3.21
IDQM	11	11	1441	0.230814	0.202118	5.79E-05	3.15
EFDM	15	15	1441	0.387854	2.111671	5.79E-05	2.74
IFDM	15	15	1441	0.385624	2.025882	5.79E-05	4.94
IDQM	15	15	1441	0.102394	0.607528	5.79E-05	4.62
EFDM	21	21	1441	0.386412	2.929844	5.79E-05	2.93
IFDM	21	21	1441	0.385011	2.855634	5.79E-05	14.15
IDQM	21	21	1441	0.125784	1.251069	5.79E-05	13.88
EFDM	31	31	1441	0.345617	3.752292	5.79E-05	3.54
IFDM	31	31	1441	0.344781	3.690951	5.79E-05	107.83
IDQM	31	31	1441	0.159122	2.00397	5.79E-05	106.42
EFDM	11	11	721	0.330592	1.198831	1.16E-04	1.55
IFDM	11	11	721	0.324565	1.129726	1.16E-04	1.82
IDQM	11	11	721	0.231105	0.201668	1.16E-04	1.86
EFDM	15	15	721	0.387118	2.084711	1.16E-04	1.61
IFDM	15	15	721	0.382576	1.903897	1.16E-04	2.71
IDQM	15	15	721	0.100153	0.576176	1.16E-04	2.75
EFDM	21	21	721	0.385914	2.904453	1.16E-04	1.79
IFDM	21	21	721	0.383023	2.744439	1.16E-04	7.42
IDQM	21	21	721	0.123556	1.163831	1.16E-04	7.11
EFDM	31	31	721	-	-	-	-
IFDM	31	31	721	0.343561	3.593802	1.16E-04	54.42
IDQM	31	31	721	0.157785	1.914814	1.16E-04	54.66

5.References

1. *Adak, M., 2020. Comparison of explicit and implicit finite difference schemes on diffusion equation. In Mathematical Modeling and Computational Tools: ICACM 2018, Kharagpur, India, November 23–25 (pp. 227-238). Springer Singapore.*
2. *Akman, M. 2003. Differential quadrature method for time-dependent diffusion equation, Middle East Technical University, M.Sc. Thesis.*
3. *Ameli, A.A. and Abedini, M.J., 2016. Performance assessment of low-order versus high-order numerical schemes in the numerical simulation of aquifer flow. Hydrology Research, 47(6), pp.1104-1115.*
4. *Baghlani, A. and Nikzad, J., 2018. Differential quadrature method for two-dimensional unsaturated water flow. Scientia Iranica, 25(1), pp.65-73.*
5. *Bahrami S., Ardejani F ,Doulati Aslani S., Baaf E. 2014. Numerical modelling of the groundwater infow to an advancing open pit mine: Kolahdarvazeh pit, Central Iran. Environmental Monitoring and Assessment. 186 (12): 8573-8585.*
6. *Behroozi, A.M. and Vaghefi, M., 2022. Thin Plates Spline Based Differential Quadrature for numerical solution of groundwater flow. Engineering Computations, 39(6), pp.2194-2208.*
7. *Bellman R., Casti J. 1971. Differential quadrature and long-term integration. Journal of Mathematical Analysis and Applications. 34(2): 235-238.*
8. *Bert C. W., Malik M. 1996. The differential quadrature method for irregular domains and application to plate vibration. International Journal of Mechanical Sciences. 38(6): 589-606.*
9. *Chan Y. L., Shen L. H., Wu C. T., Young D. L. 2014. A novel upwind-based local radial basis function differential quadrature method for convection-dominated flows. Computers & Fluids. 89: 157–166.*
10. *Chen, W. and Zhong, T., 1997. The study on the nonlinear computations of the DQ and DC methods. Numerical Methods for Partial Differential Equations: An International Journal, 13(1), pp.57-75.*
11. *Ciftci, E. (2017). Modelling coupled density-dependent flow and solute transport with the differential quadrature method. Geosciences Journal, 21, 807-817.*
12. *Fo, B., Xu, R., Xi, J., Lu, Y., Song, X., Cai, J. and Gu, Z., 2023. Numerical simulation of fine particle liquid–solid flow in porous media based on LBM-IBM-DEM. The Canadian Journal of Chemical Engineering, 101(6), pp.3576-3591.*
13. *Gharehbaghi A. 2016. Explicit and Implicit Forms of Differential Quadrature Method for Advection–Diffusion Equation with Variable Coefficients in Semi-infinite Domain. Journal of Hydrology. 541: 935-940. http://dx.doi.org/10.1016/j.jhydrol.2016.08.002*
14. *Gharehbaghi A. 2017. Third- and fifth-order finite volume schemes for advection–diffusion equation with variable coefficients in semi-infinite domain. Water and Environment Journal. 31(2): 184–193. doi: 10.1111/wej.12233*
15. *Gharehbaghi A., Kaya B., Saadatnejadgharahassanlou H. 2016. Two-dimensional bed variation models under non-equilibrium conditions in turbulent streams. Arabian Journal for Science and Engineering. 42: 999–1011. DOI 10.1007/s13369-016-2258-4*
16. *Hongen, D. and Yang, Y., 2012. Further understanding on fluid flow through multi-porous media in low-permeability reservoirs. Petroleum Exploration and Development, 39(5), pp.674-682.*
17. *Hornberger G. M., Ebert J., Remson I. 1970. Numerical Solution of the Boussinesq Equation for Aquifer-Stream Interaction. Water Resources Research. 6(2): 601-608.*
18. *Igboekwe M. U., Achi N. J. 2011. Finite difference method of modelling groundwater flow. Journal of Water Resource and Protection. 3: 192-198.*
19. *Jasak H. & Uroić T. 2020. Practical computational fluid dynamics with the finite volume method. Modeling in Engineering Using Innovative Numerical Methods for Solids and Fluids: 103-161.*
20. *Kaya B. 2010. Solution of Advection Diffusion Equation Using the Differential Quadrature Method. KSCE Journal of Civil Engineering. 14(1): 69-75.*

21. *Kaya B., Arisoy A. 2010. Differantial quadrature solution for one dimensional aquifer flow. Mathematical and Computational Applications. 16(2): 524-534.*

22. *Kovarik K, Muzik J. 2013. A meshless solution for two dimensional density-driven groundwater flow. Engineering Analysis with Boundary Elements. 37(2): 187-196.*

23. *Ku C. Y., Tsai Y. H. 2013. Solving nonlinear problems with singular initial conditions using a perturbed scalar homotopy method, International Journal of Nonlinear Sciences and Numerical Simulation. 14 (6): 367–375. DOI: 10.1515/ijnsns-2013-0029.*

24. *Kurtaran H. 2015. Large displacement static and transient analysis of functionally graded deep curved beams with generalized differential quadrature method. Composite Structures. 131: 821–831.*

25. *Larabi A., De Smedt F. 1997. Numerical solution of 3-D groundwater flow involving free boundaries by a fixed finite element method. Journal of Hydrology. 201: 161-182*

26. *Liang X., Zhang Y. K. 2011. Analytical solution for drainage and recession from an unconfined aquifer. Groundwater. 50(5): 793-798.*

27. *Liu C. Y. Ku C. Y., Huang C. C., Lin D. G., Yeih W. C. 2015. Numerical Solutions for Groundwater Flow in Unsaturated Layered Soil with Extreme Physical Property Contrasts. International Journal of Nonlinear Sciences and Numerical Simulation. 16(7-8): 325–335, DOI 10.1515/ijnsns-2015-0060*

28. *Liu S., Zhou Z., Zeng W. 2023. Simulation of Elastic Wave Propagation Based on Meshless Generalized Finite Difference Method with Uniform Random Nodes and Damping Boundary Condition. Applied Sciences, 13(3): 1312.*

29. *Mohtashami, A., Hashemi Monfared, S.A., Azizyan, G. and Akbarpour, A., 2019. Prediction of groundwater fluctuations using meshless local petrov-galerkin numerical method in a field aquifer (Birjand Aquifer). Numerical Methods in Civil Engineering, 3(4), pp.33-41.*

30. *Mohtashami, A., Monfared, S.A.H., Azizyan, G. and Akbarpour, A., 2022. Numerical simulation of groundwater in an unconfined aquifer with a novel hybrid model (case study: Birjand Aquifer, Iran). Journal of Hydroinformatics, 24(1), pp.160-178.*

31. *Nyende J., Enyogoi I., Mango J. Kasumba H. 2022. Numerical simulation of a two-dimensional groundwater pollute transport problem using incompressible steady-state Navier-Stokes equations and diffusion-convection equations. Modelling and Simulation in Engineering. 12: 2022.*

32. *Quan J. R., Chang C. T. 1989. New insights in solving distributed system equations by the quadrature method-II. Application. Computational Chemical Engineering. 13: 1017-1024.*

33. *Şeber T.A. 1964. Analysis of Pumping Tests in Wells, Turkish Geology Bulletin. 09/1-2: 71-88.*

34. *Shu C. 2000. Differential quadrature and its application in engineering. Springer Science & Business Media.*

35. *Shu C., Richards B. E. 1992. Application of generalized differential quadrature to solve two-dimensional incompressible navier-stokes equations. International Journal for Numerical Methods in Fluids. 5(1): 791-798.*

36. *Tailor K. S., Rathva G. A., Bhathawala P. H. 2013. Numerical solution of one-dimensional groundwater recharge through porous media with parabolic permeability using variational iteration method. International Journal of Advances in Engineering & Technology. 6(2): 804-811 .*

37. *Todd, D.K. and Mays, L.W., 2004. Groundwater hydrology. John Wiley & Sons.*

38. *Wang H. F. and Anderson M. P. 1995. Introduction to groundwater modeling: finite difference and finite element methods. Academic Press.*

39. *Zong Z., Zhang Y. 2009. Advanced Differential Quadrature Methods, first ed. CRC Press, Taylor and Francis Group, New York.*

40. *Zolotukhin, A.B. and Gayubov, A.T., 2019, November. Machine learning in reservoir permeability prediction and modelling of fluid flow in porous media. In IOP Conference Series: Materials Science and Engineering (Vol. 700, No. 1, p. 012023). IOP Publishing.*